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On a Theorem of Thompson and Walsh*

MISHAEL ZEDEK

Department of Mathematics, University of Maryland College Park, Maryland 20742

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DEDICATED TO MY TEACHER AND FRIEND PROFESSOR J. L. WALSH

1. In 1964 Maynard Thompson and J.L. Walsh [3] proved an extension of what has become widely known as Walsh's theorem. It is appropriate, on the occasion of Professor Walsh's seventy-fifth birthday, to come back to this subject. Thompson and Walsh's result reads as follows: Let D be the interior of a Jordan curve B. Let the function $w = f(z)$ be analytic in D, continuous in $D + B$, and map D onto some set Δ in the w-plane. Then there exists a sequence of polynomials $\{p_n(z)\}$ such that $p_n(z)$ takes on $D + B$ only values in Δ, and $p_n(z)$ converges uniformly to $f(z)$ in $D + B$.

This theorem also extended results of Carleman [1] and Farrell [2] on the approximation in a region D of an analytic function whose modulus is bounded by M, by polynomials whose moduli are, too, bounded by M in D.

The purpose of this note is to show that in some cases one can impose further restrictions on the sequence of approximating polynomials, requiring them to behave, in a certain sense, even more like the given analytic function than is the case in the above theorem of Thompson and Walsh.

2. Using the notation $f(S) = \{w \mid w = f(z), z \in S\}$, we shall prove

THEOREM 1. *Suppose $F(z)$ is analytic in a bounded Jordan region D and continuous in the closure $\bar{D}$ of D. Let D_1 be a Jordan subregion of D and suppose that there is a univalent function $\zeta = g(z)$ which maps D and D_1, respectively, onto Jordan regions D^* and D_1^* which are co-starshaped,* i.e., *starshaped with respect to the same point ζ_0. Then there exists a sequence of polynomials $\{p_k(z)\}$ such that*

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(a) *$p_k(z)$ converges uniformly to $F(z)$ in $\bar{D}$;*

(b) *$p_k(\bar{D}) \subset F(D)$ and $p_k(\bar{D}_1) \subset F(D_1)$.*

Remarks. (1) The hypothesis concerning $g(z)$ is trivally fulfilled if D and D_1 are co-starshaped. On the other hand, it can be shown that not every pair of bounded Jordan regions D and D_1, with $D_1 \subset D$, can be mapped by a univalent function onto a pair of co-starshaped regions.

(2) The method of proof can be generalized to a finite nested sequence $D_1 \supset D_2 \supset \cdots \supset D_n$, all Jordan subregions of D, provided all are sharshaped with respect to a common center or else provided they can be mapped univalently onto such a co-starshaped sequence.

(3) For an infinite sequence $\{D_i\}_{i=1}^{\infty}$ of co-starshaped Jordan regions, the approximating polynomials $\{p_k(z)\}$ can be required to satisfy $p_k(\bar{D}_i) \subset F(D_i)$ for $k \geqslant i$ and all i.

3. *Proof.* Without loss of generality, we may assume $\zeta_0 = 0$. The inverse function $z = h(\zeta)$ of $\zeta = g(z)$ is continuous in $\bar{D}^*$, since the boundaries of D and of D^* are Jordan curves. Therefore $H(\zeta) = F(h(\zeta))$ is analytic in D^*, continuous in $\bar{D}^*$ and maps D^* and $D_1{}^*$ onto $F(D)$ and $F(D_1)$, respectively.

Now, given any positive number ϵ, there exists a number $\delta(\epsilon) > 0$ such that for z_1 and z_2 in $\bar{D}$, $|z_1 - z_2| < \delta(\epsilon)$ implies $|F(z_1) - F(z_2)| < \epsilon$. Also, for any positive number δ, there exists a natural number $N(\delta)$ such that for all ζ in $\bar{D}^*$, $n \geqslant N(\delta)$ implies $|h(n\zeta/(n+1)) - h(\zeta)| < \delta$. Hence, if we denote $n_k = N(\delta(1/k))$, we may write $|F[h(g(z))] - F[h(n_k g(z)/(n_k+1))]| < 1/k$, or

$$|F(z) - F[h(n_k g(z)/(n_k+1))]| < 1/k \qquad \text{for} \quad z \in \bar{D}. \tag{1}$$

Let us denote $H_k(\zeta) = H(n_k\zeta/(n_k+1))$ for $k = 1, 2, \ldots$. Clearly, $H_k(\zeta)$ is defined in the closed Jordan region $\bar{D}_k{}^*$, where $D_k{}^* = \{\zeta \mid n_k\zeta/(n_k+1) \in D^*\}$, and $\bar{D}^* \subset D_k{}^*$. Thus $H_k(\bar{D}^*) \subset H_k(D_k{}^*) = F(D)$ and similarly $H_k(\bar{D}_1{}^*) \subset F(D_1)$.

Let d_k denote the positive distance between $H_k(\bar{D}^*)$ and the boundary of $F(D)$ and let d_k' denote the positive distance between $H_k(\bar{D}_1{}^*)$ and the boundary of $F(D_1)$. Since $H_k(\zeta)$ is analytic in D^* and continuous in $\bar{D}^*$, and since $h(\zeta)$ is univalent in $\bar{D}^*$, we can apply a theorem of Walsh [4, p. 435] to obtain a polynomial $p_k(z)$ such that

$$|H_k(\zeta) - p_k(h(\zeta))| < \min\{1/k, d_k, d_k'\} \qquad \text{for} \quad \zeta \in \bar{D}^*. \tag{2}$$

From (2) follows by substitution

$$|F[h(n_k g(z)/(n_k+1))] - p_k(z)| < 1/k \qquad \text{for} \quad z \in \bar{D}. \tag{3}$$

Combining (1) and (3), we have

$$|F(z) - p_k(z)| < 2/k \qquad \text{for} \quad z \in \bar{D} \tag{4}$$

which proves part (a) of our theorem.

To verify part (b), we observe that inequality (2) implies that $p_k(h(\bar{D}^*)) = p_k(\bar{D}) \subset F(D)$ and also $p_k(h(\bar{D}_1^*)) = p_k(\bar{D}_1) \subset F(D_1)$.

4. In the same paper [3], Thompson and Walsh have proved the following variation of their theorem quoted in the first paragraph of Section 1: Let D be a bounded region whose boundary is also the boundary of an unbounded region K. Let $w = F(z)$ be analytic in D and map it onto a set Δ of the w-plane. Then there exists a sequence of polynomials $\{p_n(z)\}$ such that $p_n(z)$ takes on $\bar{D}$ only values in Δ and converges to $f(z)$ in D, uniformly on closed subsets.

This result can be somewhat sharpened as follows:

THEOREM 2. *With the notations and assumptions of the previous paragraph, for any point z_0 in D, there exists a sequence of polynomials $\{p_n(z)\}$ such that*

(a) *$p_n(z)$ converges to $F(z)$ in D, uniformly on compact subsets;*
(b) *$p_n(z)$ takes on $\bar{D}$ only values in Δ, $n = 1, 2, \ldots$.*
(c) *$p_n(z_0) = F(z_0)$, $n = 1, 2, \ldots$.*

The proof of Thompson and Walsh (which will be modified only slightly) is based on the construction of a sequence of one-to-one mapping functions $\{\psi_n(z)\}$ which map a sequence of regions $\{G_n\}$ conformally onto D. The sequence $\{G_n\}$ is such that $D \subset G_n \subset \bar{G}_n \subset G_{n-1}$ for $n = 2, 3, \ldots$ and such that no point of K belongs to all the G_n.

We remark that the functions $\psi_n(z)$ can be chosen so that $\psi_n(z_0) = z_0$ and $\psi_n'(z_0) > 0$. This guarantees that the $g_n(z) = F(\psi_n(z))$ will satisfy $g_n(z_0) = F(z_0)$.

The function $w = g_n(z)$ maps $\bar{D}$ onto a compact subset $\bar{\Delta}_n$ of the open set Δ. Let d_n denote the positive distance between $\bar{\Delta}_n$ and the boundary of Δ. We can find a polynomial $q_n(z)$ such that

$$|g_n(z) - q_n(z)| < \min\{1/(2n), d_n/2\} \qquad \text{for} \quad z \in \bar{D}. \tag{5}$$

Define now $p_n(z) = q_n(z) - q_n(z_0) + g_n(z_0)$. It is clear that

$$|g_n(z) - p_n(z)| < 1/n \qquad \text{for} \quad z \in \bar{D} \tag{6}$$

and that the values taken by $p_n(z)$ in $\bar{D}$ are in Δ. Also, $p_n(z_0) = F(z_0)$.

By Carathéodory's well-known result, $\lim_{n\to\infty} \psi_n(z) = z$, uniformly on compact subsets of D. Therefore $\lim_{n\to\infty} g_n(z) = \lim_{n\to\infty} F(\psi_n(z)) = F(z)$ and hence, by (6), $\lim_{n\to\infty} p_n(z) = F(z)$; in both cases the convergence is uniform on compact subsets of D

It is an open question whether the single interpolation condition (c) in Theorem 2 can be replaced by several such conditions without affecting the requirement that the $p_n(z)$ take only values in Δ. (Compare Walsh [5, p. 310].)

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